Hypothesizing that the INS undergoes a 180 deg heading change, then

$$D_L^b = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \quad (13)$$

thus, using Eqs. (12) and (13) in Eq. (10) yields the following east axis equation

$$\Delta \dot{V}_{EL} = -\frac{\nabla_{Eb}}{g} g + \left(\begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \nabla_{Nb} \\ \nabla_{Eb} \\ \nabla_{Db} \end{bmatrix} \right)_{EL} \quad (14)$$

which can be written as

$$\Delta \dot{V}_{EL} = -2\nabla_{Eb} \quad (15a)$$

Using Eq. (12), this can be equivalently written as

$$\Delta \dot{V}_{EL} = -2g\phi_{NL}(0) \quad (15b)$$

Equations (15a) and (15b) express the heading sensitivity phenomenon in strapdown INS.

The preceding analysis can be illustrated graphically as follows. Figure 1 presents the goemetry at the end of the self-alignment stage while Fig. 2 presents the geometry after a 180 deg heading change. In this special example, the body axes b are chosen to coincide with the local level north-pointing axes L, thus $E_b \equiv E_L$; however, the *computed* local level north-pointing east, E_c, and down, D_c, axes are tilted by the misalignment angle $\phi_N(0)$ about the north axis. Therefore, when the *computed* transformation matrix, which transforms vector from the b to the L coordinate system, is used to transform the vector of accelerometer biases from the b to the L system, the result is the vector ∇_{Eb} along the computed east axis, E_c. As is well known, in the self-alignment stage the tilt angle is determined computationally in such a way as to satisfy Eq. (12). Indeed, the geometry of Fig. 1 expresses the equality in magnitude of the two opposing vectors $g\phi_N(0)$ and ∇_{Eb} such that

$$-g\phi_N(0) + \nabla_{Eb} = 0 \quad (16)$$

from which Eq. (12) stems. (Note that the sensed gravity vector is interpreted by the accelerometer as an acceleration in a direction opposite to the gravity vecotr. For this reason $\bar{g}$ is plotted upwards). When the inertial measuring unit of the strapdown INS is now rotated in azimuth (about the down D axis, not shown in the figures) at a 180 deg angle, this rotation is picked up by the strapdown down gyro and fed into the strapdown computer which computes a new transformation matrix. Due to the initial misalignment error, the newly computed transformation matrix implies that the body east axis, E_{bc}, differs from the true body east axis, E_b, by the same initial misalignment error $\phi_N(0)$. It also implies that the accelerometer which points in the E_b direction measures along the E_{bc} axis the quantity $g\phi_N(0)$ in addition to the accelerometer bias which is equal in magnitude to ∇_{Eb}. That is, it assumes that f_{Eb}, the specific force measured in the body east direction, is

$$f_{Eb} = \nabla_{Eb} + g\phi_N(0) \quad (17)$$

and then since D_L^{bc}, the transformation matrix from the bc to the L coordinate system, is as follows

$$D_L^{bc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -\phi_N(0) \\ 0 & -\phi_N(0) & 1 \end{bmatrix} \quad (18)$$

then when the vector of specific forces whose east value is f_{Eb} of Eq. (17) is transformed by D_L^b into the L system the result to the first order is

$$\Delta \dot{V}_{EL} = -[\nabla_{Eb} + g\phi_N(0)] \quad (19)$$

or by using Eq. (16), Eqs. (15) are obtainable.

Although a special case was used to demonstrate our argument, one can easily use any initial orientation of the body axes (i.e., not necessarily a coincidence with the L system) and any azimuth change (i.e., not necessarily 180 deg change). The argumentation will then be more involved, but the conclusions will be identical.

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The General Class of Optimal Proportional Navigation

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Introduction

PROPORTIONAL navigation (PN) is a well-known homing intercept guidance law whose performance has been studied intensively.[1,2] Theoretical and practical experience with proportional navigation has shown that the navigation ratio N' is one of the most important design parameters for PN guidance schemes to date. The ratio N' is, in practice, held fixed with acceptable values ($3 < N' < 5$) determined by noise, radome, and target maneuver considerations.[2] With the appli-

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cation of modern control theory to the field of guidance laws,[3-6] this basic role of the navigation ratio has been abandoned, since the resulting optimal gains have been shown to be restricted to just one special navigation ratio $N'=3$ (or asymptotically 3 when N' is time-varying). To date, no straightforward application of modern control theory has found reasonable weighting combinations that result in navigation ratios other than 3. This has led Kreindler[7] to solve the inverse problem. The result is time-varying weighting matrices which are difficult to interpret physically and have to be checked subsequently for positive semidefiniteness. All previous applications used two linearized kinematic states (position and velocity differences of the two bodies) for the guidance problem formulation. The two kinematic states could not reasonably be weighted independently of each other except at impact,[7] and thus the standard regulator format with running cost on the state has been regarded as inappropriate[8] for guidance optimization. The conclusion is that the familiar two-state, fixed-terminal time formulation has to be abandoned.

This Note presents a simple one-state, open-terminal time formulation of the kinematic problem, producing the general guidance law with the set of optimal PN ratios $N' \geq 3$ by straightforward application of the linear quadratic optimization theory. The navigation ratios $N'>3$ are obtained by weighting the running cost of a state variable denoted as kinematic guidance error. The performance index does not contain the terminal miss distance explicitly; however, it takes into account the effect of seeker blinding shortly before impact.

The relationship between the guidance error running cost weighting q and the navigation ratio N' is established by

$$N' = \left(3+\sqrt{4q+9}\right)/2$$

This simple relationship may serve to introduce both the standard regulator format and the navigation ratio N' into the modern controller design process.

Problem Statement

Consider an interceptor moving with constant velocity v toward a collision point with a heading error $\Delta\gamma$ from the controlled collision course γ_c, so that

$$\Delta\gamma = \gamma_c - \gamma \tag{1}$$

To reach the collision point on a circular trajectory with constant lateral control, the turning rate of the interceptor without time lag is given by

$$\dot{\gamma} = 2(\Delta\gamma/\tau) \tag{2}$$

The heading error $\Delta\gamma$ divided by the time to go τ is a measure for the interceptor's remaining control requirement for the subsequent time up to interception.

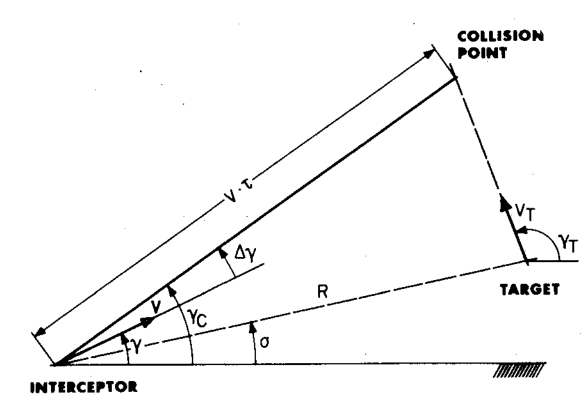

Fig. 1 Homing geometry.

For an interceptor subject to a final blind period τ_f, let the optimization interval (t_0, t_f) end at a time to go τ_f prior to impact. At this final time, the quantity $x_f = \Delta\gamma_f/\tau_f$, kept small by means of a quadratic terminal cost criterion, determines the interceptor's ability to hit the collision point. This in turn gives credibility to an optimization approach which uses the final time to go τ_f as a parameter. The homing geometry is depicted in Fig. 1.

With the quantity $x=\Delta\gamma/\tau$ taken as a state variable, consider the running cost

$$\int_{t_0}^{t_f} q x^2 \, \mathrm{d}t$$

in a quadratic performance index. With the definition of x, the weighting q allows penalizing all in-flight deviations $\Delta\gamma$ from the collision course in such a way that future demands on the acceleration capability of the interceptor are kept small. The quantity x is denoted as the kinematic guidance error. Differentiation of x yields

$$\dot{x} = \frac{\dot{\gamma}_c - \dot{\gamma}}{\tau} + \frac{\Delta\gamma}{\tau^2} \tag{3}$$

Since the interceptor does not travel on a collision course trajectory, the collision course γ_c changes during flight. The turning rate of the collision course is derived from the geometry as $\dot{\gamma}_c = \Delta\gamma/\tau$, and Eq. (3) becomes

$$\dot{x} = \frac{2}{\tau}x - \frac{1}{\tau}\dot{\gamma} \tag{4}$$

which is a scalar state formulation of the interception problem of the form

$$\dot{x} = Ax + Bu$$

with $u=\dot{\gamma}$ as the control variable. This state formulation so far is independent of the target motion.

The scalar linear differential equation (4) is exclusively specialized to the homing guidance problem. The reduction in order, relative to the familiar two-state formulations,[3,4] goes along with acceptance that the rendezvous and soft landing problems[3] are no longer candidate subjects of optimization. For the interceptor guidance, however, it is advantageous to apply Eq. (4) for the sake of transparency.

The problem now is to minimize the quadratic performance index

$$J = \frac{1}{2} f x^2(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [qx^2 + ru^2]\, \mathrm{d}t \tag{5}$$

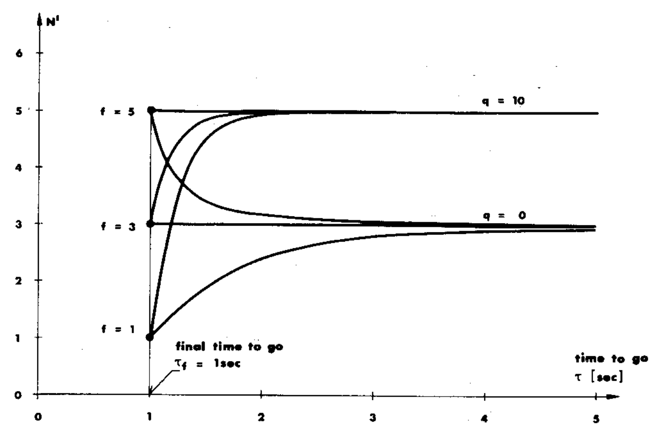

Fig. 2 Optimal PN ratios.

with weighting scalars f, q, and r by solving the Riccati differential equation

$$\dot{K} = -2K\frac{2}{\tau} + K^2\left(\frac{1}{\tau^2}\right)\frac{1}{r} - q \tag{6}$$

with the boundary condition

$$K(t_f) = f \tag{7}$$

Derivation of the Guidance Law

Substituting

$$K = z\tau \qquad \dot{K} = \dot{z}\tau - z$$

we obtain, with the identity $\mathrm{d}t = -\mathrm{d}\tau$, the differential equation

$$r\frac{\mathrm{d}z}{z^2 - 3rz - qr} = -\frac{\mathrm{d}\tau}{\tau} \tag{8}$$

which is integrated and transformed to yield

$$K = \tau\left(\frac{3}{2} + \frac{d}{2}\frac{\tau^{d/r} + c}{\tau^{d/r} - c}\right) \tag{9}$$

with c an integration constant.

In Eq. (9) the abbreviation

$$d = \sqrt{4qr + 9r^2} \tag{10}$$

is used.

Now let t_f be the final time of optimization and accordingly let τ_f be the final (nonzero) time to go; then boundary condition (7) determines the integration constant $c = e\tau_f^{d/r}$ and Eq. (9) becomes

$$K = \frac{\tau}{2}(d + 3r)\frac{\left(\dfrac{\tau}{\tau_f}\right)^{d/r} + \dfrac{d-3r}{d+3r}e}{\left(\dfrac{\tau}{\tau_f}\right)^{d/r} - e} \tag{11}$$

where the abbreviation

$$e = \frac{f(3r + d) - 2f}{f(3r - d) - 2f} \tag{12}$$

is a function of the weightings f, q, and r.

The optimal closed-loop guidance law now can be expressed as the result of the scalar equation

$$u = -\frac{1}{r}BKx \tag{13}$$

which becomes

$$\dot{\gamma} = \frac{d + 3r}{2r}\frac{\left(\dfrac{\tau}{\tau_f}\right)^{d/r} + \dfrac{d-3r}{d+3r}e}{\left(\dfrac{\tau}{\tau_f}\right)^{d/r} - e}\left(\frac{\Delta\gamma}{\tau}\right) \tag{14}$$

The collision course guidance law [Eq. (14)] has the form

$$\dot{\gamma} = N'\left(\frac{\Delta\gamma}{\tau}\right) \tag{15}$$

with N' being a time-varying navigation ratio.

For the special condition of constant closing velocity $v_c = R/\tau$ and for small heading errors $\Delta\gamma \approx \sin(\Delta\gamma)$, it can be shown easily that by the use of the collision course condition

$$v_T\sin(\gamma_T - \sigma) = v\sin(\gamma_c - \sigma)$$

the state variable $\Delta\gamma/\tau$ is related to the line-of-sight rate $\dot{\sigma}$ by

$$\frac{\Delta\gamma}{\tau} = \frac{v_c}{v\cos(\gamma - \sigma)}\dot{\sigma} \tag{16}$$

Thus, the PN-type guidance law is given by

$$\dot{\gamma} = N'\frac{v_c}{v\cos(\gamma - \sigma)}\dot{\sigma} \tag{17}$$

The navigation ratio N' can be further investigated to interpret its dependence on the guidance law weighting parameters f, q, r, and the final blind period τ_f.

Discussion of the Navigation Ratio

Let the constraint on the control effort be normalized by setting $r = 1$. The ratio now becomes

$$N' = \frac{d+3}{2}\frac{\left(\dfrac{\tau}{\tau_f}\right)^{d} + \dfrac{d-3}{d+3}e}{\left(\dfrac{\tau}{\tau_f}\right)^{d} - e} \tag{18}$$

with the abbreviations d and e denoting

$$d = \sqrt{4q + 9} \qquad e = \frac{\tau_f(3 + d) - 2f}{\tau_f(3 - d) - 2f} \tag{19}$$

Let us now look at the nature of N' more closely. First let $\tau_f = 0$ in order to produce the guidance law for homing up to interception. In this case, the time-varying function drops out and Eq. (18) converges to the constant ratio

$$N' = \frac{d+3}{2} = \frac{3 + \sqrt{4q + 9}}{2} \tag{20}$$

The ratio N' depends only on the running cost weighting q, thereby producing the set of constant navigation ratios $N' \geq 3$.

For the case of guidance to handover to a final blind period τ_f, we have $\tau_f > 0$, and Eq. (20) gives the asymptotes of nonstationary N'. For $\tau \to \tau_f$, the final weighting f governs the navigation ratio, and at the time to go at handover, $\tau = \tau_f$, the ratio is determined by

$$N'(\tau_f) = f/\tau_f \tag{21}$$

For the special choice of final weighting

$$f = \tau_f\frac{\sqrt{4q + 9} + 3}{2} \tag{22}$$

the time-varying function in Eq. (18) cancels, and again results in the constant gain equation (20). This shows that the classical PN-guidance law is just one candidate for optimal guidance to handover to a subsequent phase which is not treated here.

The navigation ratio N' as computed from Eq. (18) is plotted parametrically in Fig. 2 for some typical design values and a final blind interval of 1 s.

Conclusion

The kinematic guidance problem has been reformulated and a general optimal guidance law has been derived using the linear quadratic controller design theory. The approach differs from foregoing work in three points: the use of a scalar state formulation which is valid near impact, the use of an optimization interval for which the terminal time is an open design parameter, and the use of the standard regulator format with running cost on the system state. The resulting kinematic guidance law is more general than previous results, since it produces the whole range of classical proportional navigation ratios on the one hand, and, on the other hand, it shows that the proportional navigation law (up to interception) is just an extreme case of guidance to handover to a final blind period. Final blind periods are significant in practice due to seeker measurement corruptions and image explosions near impact. Thus the practical problem is often to minimize a performance criterion continuously during flight and at the final blind time, and not exclusively at impact—a fact which in most theoretical assessments is not considered.

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An Analytic Solution for the State Trajectories of a Feedback Control System

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Introduction

As part of the normal process of a control system design, the analyst frequently is interested in determining the state trajectories for the controlled system. In practice, this process is straightforward, since the feedback form of the control can be introduced in the equation of motion and numerically integrated. Nevertheless, this process can be computationally intensive, if either time-varying control gains are used, or if small integration step sizes are required by the presence of high-frequency system dynamics.

In an effort to overcome the computational difficulties listed above, we present in this Note a change of variables for the closed-loop system dynamics equation, which permits a closed-form expression to be obtained for the state trajectories.

Optimal Control Problem

The fixed time linear optimal control problem is formulated by finding the control inputs $u(t)$ to minimize

$$J=\frac{1}{2}x_f^TF^TSFx_f+\frac{1}{2}\int_{t_0}^{t_f}(x^TF^TQFx+u^TRu)\,dt \tag{1}$$

for the system

$$\dot{x}=Ax+Bu, \quad \text{given} \quad x(t_0) \tag{2}$$

$$y=Fx \tag{3}$$

where x is the state, u is the control, A is the system dynamics matrix, B is the control influence matrix, F is the measurement influence matrix, Q is the output weighting matrix, R is the control weight matrix, and S is the terminal output weight matrix.

As shown in Ref. 1, the optimal control is given by

$$u(t)=-R^{-1}B^TP(t)x(t) \tag{4}$$

where P is the solution to the differential matrix Riccati equation

$$\dot{P}=-PA-A^TP+PBR^{-1}B^TP-F^TQF; \quad P(t_f)=F^TSF \tag{5}$$

Upon introducing Eq. (4) into Eq. (2), the standard closed-loop system dynamics equation follows as

$$\dot{x}(t)=[A-BR^{-1}B^TP(t)]x(t); \quad x_0=x(t_0) \tag{6}$$

To simplify the solution to Eq. (6), we introduce the following closed-form solution for $P(t)$[2-8] in Eq. (5):

$$P(t)=P_{ss}+Z^{-1}(t) \tag{7}$$

where P_{ss} is the solution to the algebraic Riccati equation[9,10]

$$-PA-A^TP+PBR^{-1}B^TP-F^TQF=0 \tag{8}$$

and $Z(t)$ is a matrix function which is to be determined.

Upon introducing Eq. (7) and its time derivative into Eq. (5), the linear constant coefficient matrix differential equation for $Z(t)$ follows as

$$\dot{Z}=\bar{A}Z+Z\bar{A}^T-BR^{-1}B^T; \quad Z(t_f)=(F^TSF-P_{ss})^{-1} \tag{9}$$

from which it follows that the solution for $Z(t)$[11,12] is given by

$$Z(t)=Z_{ss}+e^{\bar{A}(t-t_f)}[Z(t_f)-Z_{ss}]e^{\bar{A}^T(t-t_f)} \tag{10}$$

where $\bar{A}=A-BR^{-1}B^TP_{ss}$ is the steady-state closed-loop system matrix, $e^{(\cdot)}$ is the exponential matrix, and Z_{ss} satisfies the algebraic Lyapunov equation[10,13-15]

$$\bar{A}Z_{ss}+Z_{ss}\bar{A}^T=BR^{-1}B^T$$

Substituting Eq. (7) into Eq. (6) yields the modified form of the closed-loop system dynamics equation

$$\dot{x}(t)=[\bar{A}-BR^{-1}B^TZ^{-1}(t)]x(t); \quad x_0=x(t_0) \tag{11}$$

where we observe that the equation above is nonautonomous.

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